

Accuracy on eigenvalues for a Schrödinger operator with a degenerate potential in the semi-classical limit

Abderemane MORAME¹ and Françoise TRUC²

¹ Université de Nantes, Faculté des Sciences, Dpt. Mathématiques,
UMR 6629 du CNRS, B.P. 99208, 44322 Nantes Cedex 3, (FRANCE),
E.Mail: morame@math.univ-nantes.fr

² Université de Grenoble I, Institut Fourier,
UMR 5582 CNRS-UJF, B.P. 74,
38402 St Martin d'Hères Cedex, (France),
E.Mail: Françoise.Truc@ujf-grenoble.fr

¹

Abstract

We consider a semi-classical Schrödinger operator $-h^2\Delta + V$ with a degenerate potential $V(x,y) = f(x)g(y)$.

g is assumed to be a homogeneous positive function of m variables and f is a strictly positive function of n variables, with a strict minimum.

We give sharp asymptotic behaviour of low eigenvalues bounded by some power of the parameter h , by improving Born-Oppenheimer approximation.

1 Introduction

In our paper [MoTr] we have considered the Schrödinger operator on $L^2(\mathbb{R}_x^n \times \mathbb{R}_y^m)$

$$\widehat{H}_h = h^2 D_x^2 + h^2 D_y^2 + f(x)g(y) \quad (1.1)$$

with $g \in C^\infty(\mathbb{R}^m \setminus \{0\})$ homogeneous of degree $a > 0$,

$$g(\mu y) = \mu^a g(y) > 0, \quad \forall \mu > 0 \text{ and } \forall y \in \mathbb{R}^m \setminus \{0\}. \quad (1.2)$$

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$h > 0$ is a semiclassical parameter we assume to be small.

We have investigated the asymptotic behavior of the number of eigenvalues less than λ of \widehat{H}_h ,

$$N(\lambda, \widehat{H}_h) = \text{tr}(\chi_{]-\infty, \lambda[}(\widehat{H}_h)) = \sum_{\lambda_k(\widehat{H}_h) < \lambda} 1 . \quad (1.3)$$

($\text{tr}(P)$ denotes the trace of the operator P).

If P is a self-adjoint operator on a Hilbert space \mathcal{H} , we denote respectively by $\text{sp}(P)$, $\text{sp}_{\text{ess}}(P)$ and $\text{sp}_d(P)$ the spectrum, the essential spectrum and the discrete spectrum of P .

When $-\infty < \inf \text{sp}(P) < \inf \text{sp}_{\text{ess}}(P)$, we denote by $(\lambda_k(P))_{k>0}$ the increasing sequence of eigenvalues of P , repeated according to their multiplicity:

$$\text{sp}_d(P) \bigcap]-\infty, \inf \text{sp}_{\text{ess}}(P)[= \{\lambda_k(P)\} . \quad (1.4)$$

In this paper we are interested in a sharp estimate for some eigenvalues of \widehat{H}_h . We make the following assumptions on the other multiplicative part of the potential:

$$\begin{aligned} f &\in C^\infty(\mathbb{R}^n), \forall \alpha \in \mathbb{N}^n, (|f(x)| + 1)^{-1} \partial_x^\alpha f(x) \in L^\infty(\mathbb{R}^n) \\ 0 &< f(0) = \inf_{x \in \mathbb{R}^n} f(x) \\ f(0) &< \liminf_{|x| \rightarrow \infty} f(x) = f(\infty) \\ \partial^2 f(0) &> 0 \end{aligned} \quad (1.5)$$

$\partial^2 f(a)$ denotes the hessian matrix:

$$\partial^2 f(a) = \left(\frac{\partial^2}{\partial x_i \partial x_j} f(a) \right)_{1 \leq i, j \leq n} .$$

By dividing \widehat{H}_h by $f(0)$, we can change the parameter h and assume that

$$f(0) = 1 . \quad (1.6)$$

Let us define : $\hbar = h^{2/(2+a)}$ and change y in $y\hbar$; we can use the homogeneity of g (1.2) to get :

$$\text{sp}(\widehat{H}_h) = \hbar^a \text{sp}(\widehat{H}^\hbar) , \quad (1.7)$$

with $\hat{H}^\hbar = \hbar^2 D_x^2 + D_y^2 + f(x)g(y) = \hbar^2 D_x^2 + Q(x, y, D_y) :$

$$Q(x, y, D_y) = D_y^2 + f(x)g(y).$$

Let us denote the increasing sequence of eigenvalues of $D_y^2 + g(y)$, (on $L^2(\mathbb{R}^m)$), by $(\mu_j)_{j>0}$.

The associated eigenfunctions will be denoted by $(\varphi_j)_j$:

$$\begin{aligned} D_y^2 \varphi_j(y) + g(y) \varphi_j(y) &= \mu_j \varphi_j(y) \\ \langle \varphi_j | \varphi_k \rangle &= \delta_{jk} \end{aligned} \quad (1.8)$$

and $(\varphi_j)_j$ is a Hilbert base of $L^2(\mathbb{R}^m)$.

By homogeneity (1.2) the eigenvalues of $Q_x(y, D_y) = D_y^2 + f(x)g(y)$, on $L^2(\mathbb{R}^m)$, for a fixed x , are given by the sequence $(\lambda_j(x))_{j>0}$, where : $\lambda_j(x) = \mu_j f^{2/(2+a)}(x)$.

So as in [MoTr] we get :

$$\hat{H}^\hbar \geq [\hbar^2 D_x^2 + \mu_1 f^{2/(2+a)}(x)]. \quad (1.9)$$

This estimate is sharp as we will see below.

Then using the same kind of estimate as (1.9), one can see that

$$\inf sp_{ess}(\hat{H}^\hbar) \geq \mu_1 f^{2/(2+a)}(\infty). \quad (1.10)$$

We are in the Born-Oppenheimer approximation situation described by A. Martinez in [Ma] : the "effective" potential is given by $\lambda_1(x) = \mu_1 f^{2/(2+a)}(x)$, the first eigenvalue of Q_x , and the assumptions on f ensure that this potential admits one unique and nondegenerate well $U = \{0\}$, with minimal value equal to μ_1 . Hence we can apply theorem 4.1 of [Ma] and get :

Theorem 1.1 *Under the above assumptions, for any arbitrary $C > 0$, there exists $h_0 > 0$ such that, if $0 < \hbar < h_0$, the operator (\hat{H}^\hbar) admits a finite number of eigenvalues $E_k(\hbar)$ in $[\mu_1, \mu_1 + C\hbar]$, equal to the number of the eigenvalues e_k of $D_x^2 + \frac{\mu_1}{2+a} < \partial^2 f(0) x$, $x >$ in $[0, +C]$ such that :*

$$E_k(\hbar) = \lambda_k(\hat{H}^\hbar) = \lambda_k(\hbar^2 D_x^2 + \mu_1 f^{2/(2+a)}(x)) + \mathbf{O}(\hbar^2). \quad (1.11)$$

More precisely $E_k(\hbar) = \lambda_k(\hat{H}^\hbar)$ has an asymptotic expansion

$$E_k(\hbar) \sim \mu_1 + \hbar (e_k + \sum_{j \geq 1} \alpha_{kj} \hbar^{j/2}). \quad (1.12)$$

If $E_k(\hbar)$ is asymptotically non degenerated, then there exists a quasimode

$$\phi_k^\hbar(x, y) \sim \hbar^{-m_k} e^{-\psi(x)/\hbar} \sum_{j \geq 0} \hbar^{j/2} a_{kj}(x, y), \quad (1.13)$$

satisfying

$$\begin{aligned} C_0^{-1} &\leq \|\hbar^{-m_k} e^{-\psi(x)/\hbar} a_{k0}(x, y)\| \leq C_0 \\ \|\hbar^{-m_k} e^{-\psi(x)/\hbar} a_{kj}(x, y)\| &\leq C_j \\ \|\left(\widehat{H}^\hbar - \mu_1 - \hbar e_k - \sum_{1 \leq j \leq J} \alpha_{kj} \hbar^{j/2}\right) \\ \hbar^{-m_k} e^{-\psi(x)/\hbar} \sum_{0 \leq j \leq J} \hbar^{j/2} a_{kj}(x, y)\| &\leq C_J \hbar^{(J+1)/2} \end{aligned} \quad (1.14)$$

The formula (1.12) implies

$$\lambda_k(\widehat{H}^\hbar) = \mu_1 + \hbar \lambda_k \left(D_x^2 + \frac{\mu_1}{2+a} < \partial^2 f(0) x, x > \right) + \mathbf{O}(\hbar^{3/2}), \quad (1.15)$$

and when $k = 1$, one can improve $\mathbf{O}(\hbar^{3/2})$ into $\mathbf{O}(\hbar^2)$. The function ψ is defined by : $\psi(x) = d(x, 0)$, where d denotes the Agmon distance related to the degenerate metric $\mu_1 f^{2/(2+a)}(x) dx^2$.

2 Lower energies

We are interested now with the lower energies of \widehat{H}^\hbar . Let us make the change of variables

$$(x, y) \rightarrow (x, f^{1/(2+a)}(x)y). \quad (2.1)$$

The Jacobian of this diffeomorphism is $f^{m/(2+a)}(x)$, so we perform the change of test functions : $u \rightarrow f^{-m/(4+2a)}(x)u$, to get a unitary transformation.

Thus we get that

$$sp(\widehat{H}^\hbar) = sp(\tilde{H}^\hbar) \quad (2.2)$$

where \tilde{H}^\hbar is the self-adjoint operator on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$ given by

$$\tilde{H}^\hbar = \hbar^2 L^*(x, y, D_x, D_y) L(x, y, D_x, D_y) + f^{2/(2+a)}(x) (D_y^2 + g(y)), \quad (2.3)$$

with

$$L(x, y, D_x, D_y) = D_x + \frac{1}{(2+a)f(x)} [(yD_y) - i\frac{m}{2}] \nabla f(x).$$

We decompose \tilde{H}^\hbar in four parts :

$$\begin{aligned}\tilde{H}^\hbar &= \hbar^2 D_x^2 + f^{2/(2+a)}(x) (D_y^2 + g(y)) \\ &+ \hbar^2 \frac{2}{(2+a)f(x)} (\nabla f(x) D_x) (y D_y) \\ &+ i \hbar^2 \frac{1}{(2+a)f^2(x)} (|\nabla f(x)|^2 - f(x) \Delta f(x)) [(y D_y) - i \frac{m}{2}] \\ &+ \hbar^2 \frac{1}{(2+a)^2 f^2(x)} |\nabla f(x)|^2 [(y D_y)^2 + \frac{m^2}{4}]\end{aligned}\quad (2.4)$$

Our goal is to prove that the only significant role up to order 2 in \hbar will be played by the first operator, namely : $\tilde{H}_1^\hbar = \hbar^2 D_x^2 + f^{2/(2+a)}(x) (D_y^2 + g(y))$.

Let us denote by $\nu_{j,k}^\hbar$ the eigenvalues of the operator $\hbar^2 D_x^2 + \mu_j f^{2/(2+a)}(x)$ and by $\psi_{j,k}^\hbar$ the associated normalized eigenfunctions .

Let us consider the following test functions :

$$u_{j,k}^\hbar(x, y) = \psi_{j,k}^\hbar(x) \varphi_j(y) ,$$

where the φ_j 's are the eigenfunctions defined in (1.8); we have immediately :

$$\tilde{H}_1^\hbar(u_{j,k}^\hbar(x, y)) = \nu_{j,k}^\hbar u_{j,k}^\hbar(x, y) .$$

We will need the following lemma :

Lemma 2.1 . *For any integer N , there exists a positive constant C depending only on N such that for any $k \leq N$, the eigenfunction $\psi_{j,k}^\hbar$ satisfies the following inequalities : for any $\alpha \in \mathbb{N}^n$, $|\alpha| \leq 2$,*

$$\begin{aligned}\| \hbar_j^{|\alpha|/2} |D_x^\alpha \psi_{j,k}^\hbar| \| &< C \\ \left\| \left(\frac{\nabla f(x)}{f(x)} \right)^\alpha \psi_{j,k}^\hbar \right\| &< \hbar_j^{|\alpha|/2} C\end{aligned}\quad (2.5)$$

with $\hbar_j = \hbar \mu_j^{-1/2}$.

Proof.

Let us recall that it is well known, (see [He-Sj1]), that

$$\forall k \leq N, \quad \mu_j^{-1} \nu_{j,k}^\hbar = 1 + \mathbf{O}(\hbar_j) .$$

It is clear also that

$$[\hbar_j^2 D_x^2 + f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^\hbar] \psi_{j,k}^\hbar(x) = 0 . \quad (2.6)$$

We shall need the following inequality, that we can derive easily from (2.6) and the Agmon estimate (see [He-Sj1]) : $\forall \varepsilon \in]0, 1[$,

$$\varepsilon \int [f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^{\hbar}]_+ e^{2(1-\varepsilon)^{1/2} d_{j,k}(x)/\hbar_j} |\psi_{j,k}^{\hbar}(x)|^2 dx \leq \int [f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^{\hbar}]_- |\psi_{j,k}^{\hbar}(x)|^2 dx, \quad (2.7)$$

where $d_{j,k}$ is the Agmon distance associated to the metric $[f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^{\hbar}]_+ dx^2$.

Let us prove the lemma for $|\alpha| = 1$.

As $\int [\hbar_j^2 |D_x \psi_{j,k}^{\hbar}(x)|^2 + (f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^{\hbar}) |\psi_{j,k}^{\hbar}(x)|^2] dx = 0$,
 $\mu_j^{-1} \nu_{j,k}^{\hbar} - 1 = \mathbf{O}(\hbar_j)$, and $f^{2/(2+a)}(x) - 1 > 0$,
we get that $\hbar_j \|D_x \psi_{j,k}^{\hbar}(x)\|^2 \leq C$.

Furthermore, we use that $C^{-1} |\nabla f(x)|^2 \leq f^{2/(2+a)}(x) - 1 \leq C |\nabla f(x)|^2$,
for $|x| \leq C^{-1}$, the exponential decreasing (in \hbar_j) of $\psi_{j,k}^{\hbar}$ given by (2.7)
and the boundness of $|\nabla f(x)|/f(x)$ to get

$$\left\| \frac{|\nabla f(x)|}{f(x)} \psi_{j,k}^{\hbar}(x) \right\|^2 \leq C \int [f^{2/(2+a)}(x) - 1] |\psi_{j,k}^{\hbar}(x)|^2 dx \leq \hbar_j C.$$

Now we study the case $|\alpha| = 2$.

If $c_0 \in]0, 1]$ is large enough and $|x| \in [\hbar_j^{1/2} c_0, 2c_0]$, then we have

$$|x|^2/C \leq f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^{\hbar} \leq C|x|^2 \quad (2.8)$$

Therefore there exists $C_1 > 1$ such that $C_1^{-1} |x|^2 \leq d_{j,k}(x) \leq C_1 |x|^2$,
and then

$$|x|^2 \leq \hbar_j C e^{d_{j,k}(x)/\hbar_j}. \quad (2.9)$$

Then the inequality : $C^{-1} |x| \leq |\nabla f(x)| \leq C |x|$. together with (2.8),
(2.9) and (2.7) entail that

$$\begin{aligned} \int_{|x| \geq C_0 \hbar_j^{1/2}} \frac{|\nabla f(x)|^4}{f^4(x)} |\psi_{j,k}^{\hbar}(x)|^2 dx \\ &\leq \hbar_j C \int [f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^{\hbar}]_+ e^{d_{j,k}(x)/\hbar_j} |\psi_{j,k}^{\hbar}(x)|^2 dx \\ &\leq \hbar_j C \int [f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^{\hbar}]_- |\psi_{j,k}^{\hbar}(x)|^2 dx \\ &\leq \hbar_j^2 C. \end{aligned}$$

It remains to estimate $\hbar_j^2 \|D_x^\alpha \psi_{j,k}^{\hbar}(x)\|$ with $|\alpha| = 2$.

We use that $-\hbar_j^2 \Delta \psi_{j,k}^{\hbar}(x) = [-f^{2/(2+a)}(x) + \mu_j^{-1} \nu_{j,k}^{\hbar}] \psi_{j,k}^{\hbar}(x)$,
and that we have proved that $\|[-f^{2/(2+a)}(x) + \mu_j^{-1} \nu_{j,k}^{\hbar}] \psi_{j,k}^{\hbar}(x)\| \leq \hbar_j C$;
so $\|D_x^\alpha \psi_{j,k}^{\hbar}(x)\| \leq C/\hbar_j$ if $|\alpha| = 2$.

We will need the following result.

Proposition 2.2 *Let $V(y) \in C^\infty(\mathbb{R}^m)$ such that*

$$\begin{aligned} \exists s > 0, \quad C_0 > 0 \text{ s.t. } -C_0 + |y|^s/C_0 \leq V(y) \leq C_0(|y|^s + 1) \\ \forall \alpha \in \mathbb{N}^m, \quad (1 + |y|^{2(s-|\alpha|)})^2 \partial_y^\alpha V(y) \in L^\infty(\mathbb{R}^m). \end{aligned} \quad (2.10)$$

If $u(y) \in L^2(\mathbb{R}^m)$ and $D_y^2 u(y) + V(y)u(y) \in S(\mathbb{R}^m)$, then $u \in S(\mathbb{R}^m)$. ($S(\mathbb{R}^m)$ is the Schwartz space).

The proof comes from the fact that there exists a parametrix of $D_y^2 + V(y)$ in some class of pseudodifferential operator: see for the more general case in [Hor], or for this special case in Shubin book [Shu].

Theorem 2.3 .

Under the assumptions (1.2) and (1.5), for any fixed integer $N > 0$, there exists a positive constant $h_0(N)$ verifying : for any $\hbar \in]0, h_0(N)[$, for any $k \leq N$ and any $j \leq N$ such that

$$\mu_j < \mu_1 f^{2/(2+a)}(\infty),$$

there exists an eigenvalue $\lambda_{jk} \in \text{sp}_d(\widehat{H}^\hbar)$ such that

$$|\lambda_{jk} - \lambda_k(\hbar^2 D_x^2 + \mu_j f^{2/(2+a)}(x))| \leq \hbar^2 C. \quad (2.11)$$

Consequently, when $k = 1$, we have

$$|\lambda_{j1} - \left[\mu_j + \hbar(\mu_j)^{1/2} \frac{\text{tr}((\partial^2 f(0))^{1/2})}{(2+a)^{1/2}} \right]| \leq \hbar^2 C. \quad (2.12)$$

Proof .

The first part of the theorem will follow if we prove that :

$$\|(\widehat{H}^\hbar - \widetilde{H}_1^\hbar)(u_{j,k}^\hbar(x, y))\| = \|(\widehat{H}^\hbar - \nu_{j,k}^\hbar)u_{j,k}^\hbar(x, y)\| = \mathbf{O}(\hbar^2).$$

Let us consider a function $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi(t) = 1 \text{ if } |t| \leq 1/2 \text{ and}$$

$$\chi(t) = 0 \text{ if } |t| > 1.$$

Then $(D_y^2 + g(y))(1 - \chi(|y|))\varphi_j(y) \in S(\mathbb{R}^m)$,

and Proposition 2.2 shows that $(1 - \chi(|y|))\varphi_j(y) \in S(\mathbb{R}^m)$.

As $D_y^2 \varphi_j(y) = (\mu_j - g(y))\varphi_j(y)$, we get that

$$\forall k \in \mathbb{N}, \quad (1 + |y|)^k [|\varphi_j(y)|^2 + |D_y \varphi_j(y)|^2 + |D_y^2 \varphi_j(y)|^2] \in L^1(\mathbb{R}^m). \quad (2.13)$$

The quantity $(\hat{H}^\hbar - \tilde{H}_1^\hbar)(u_{j,k}^\hbar(x,y))$ is, by (2.4), composed of 3 parts. According to Lemma 2.1 and the estimate (2.13), the two last parts are bounded in L^2 -norm by $\hbar^2 C$, ($\mu_j \leq C$).

To obtain a bound for the first part, we integrate by parts to get that

$$\left\| \frac{\nabla f(x)}{f(x)} D_x \psi_{j,k}^\hbar \right\|^2 \leq C \left[\|D_x^2 \psi_{j,k}^\hbar\| \times \left\| \frac{|\nabla f(x)|^2}{f^2(x)} \psi_{j,k}^\hbar \right\| + \|D_x \psi_{j,k}^\hbar\| \times \left\| \frac{|\nabla f(x)|}{f(x)} \psi_{j,k}^\hbar \right\| \right],$$

and then we use again Lemma 2.1. Thus : $\left\| \frac{\nabla f(x)}{f(x)} D_x \psi_{j,k}^\hbar \right\| \leq C$.

According to estimate (2.13) we have finally $\left\| \frac{\nabla f(x)}{f(x)} D_x (y D_y) u_{j,k}^\hbar \right\| \leq C$.

3 Middle energies

We are going to refine the preceding results when $a \geq 2$ and $f(\infty) = \infty$. It is possible then to get sharp localization near the μ_j 's for much higher values of j 's. More precisely we prove :

Theorem 3.1 . We assume (1.5) with $f(\infty) = \infty$, (1.2) with $a \geq 2$ and with $g \in C^\infty(\mathbb{R}^m)$.

Let us consider j such that $\mu_j \leq \hbar^{-2}$; then for any integer N , there exists a constant C depending only on N such that, for any $k \leq N$, there exists an eigenvalue $\lambda_{jk} \in \text{sp}_d(\hat{H}^\hbar)$ verifying

$$|\lambda_{jk} - \lambda_k(\hbar^2 D_x^2 + \mu_j f^{2/(2+a)}(x))| \leq C \mu_j \hbar^2. \quad (3.1)$$

Consequently, when $k = 1$, we have

$$|\lambda_{j1} - \left[\mu_j + \hbar(\mu_j)^{1/2} \frac{\text{tr}((\partial^2 f(0))^{1/2})}{(2+a)^{1/2}} \right]| \leq C \mu_j \hbar^2. \quad (3.2)$$

Proof :

Let us define the class of symbols $S(p^s(y, \eta))$, $s \in \mathbb{R}$, with $p(y, \eta) = |\eta|^2 + g(y) + 1$.

$$q(y, \eta) \in S(p^s(y, \eta)) \quad \text{iff} \quad q(y, \eta) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m)$$

and for any α and $\beta \in \mathbb{N}^m$,

$$p^{-s}(y, \eta)(|\eta| + 1)^{-|\alpha|}(|y| + 1)^{-|\beta|} D_\eta^\alpha D_y^\beta q(y, \eta) \in L^\infty(\mathbb{R}^{2m}).$$

For such a symbol $q(y, \eta) \in S(p^s(y, \eta))$, we define the operator Q on $S(\mathbb{R}^m)$:

$$Qf(y) = (2\pi)^{-m} \int_{\mathbb{R}^{2m}} q\left(\frac{y+z}{2}, \eta\right) e^{i(y-z)\eta} f(z) dz d\eta.$$

We will say that $Q \in OPS(p^s(y, \eta))$.

It is well known, (see [Hor]) that $(D_y^2 + g(y))^s \in OPS(p^s(y, \eta))$.

As $a \geq 2$, we get that $yD_y \in OPS(p(y, \eta))$, and then that $yD_y(D_y^2 + g(y))^{-1} \in OPS(1)$.

Therefore $yD_y(D_y^2 + g(y))^{-1}$ and $(yD_y)^2(D_y^2 + g(y))^{-2}$ are bounded operator on $L^2(\mathbb{R}^m)$, and we get as a consequence the following bound :

$$\mu_j^{-1} \|yD_y \varphi_j\| + \mu_j^{-2} \|(yD_y)^2 \varphi_j\| \leq C. \quad (3.3)$$

As in the proof of Theorem 2.3, using (3.3) instead of (2.13), we get easily that

$$\|(\hat{H}^\hbar - \tilde{H}^\hbar)u_{j,k}^\hbar\| \leq C[\hbar^2 \mu_j + \hbar^3 \mu_j^{3/2}] \leq C\hbar^2 \mu_j,$$

and then Theorem 3.1 follows.

4 An application

We consider a Schrödinger operator on $L^2(\mathbb{R}_z^d)$ with $d \geq 2$,

$$P^\hbar = -\hbar^2 \Delta + V(z) \quad (4.1)$$

with a real and regular potential $V(z)$ satisfying

$$\begin{aligned} V &\in C^\infty(\mathbb{R}^d; [0, +\infty]) \\ \liminf_{|z| \rightarrow \infty} V(z) &> 0 \\ \Gamma &= V^{-1}(\{0\}) \text{ is a regular hypersurface.} \end{aligned} \quad (4.2)$$

By hypersurface, we mean a submanifold of codimension 1. Moreover we assume that Γ is connected and that there exist $m \in \mathbb{N}^*$ and $C_0 > 0$ such that for any z verifying $d(z, \Gamma) < C_0^{-1}$

$$C_0^{-1} d^{2m}(z, \Gamma) \leq V(z) \leq C_0 d^{2m}(z, \Gamma) \quad (4.3)$$

($d(E, F)$ denotes the euclidian distance between E and F).

We choose an orientation on Γ and a unit normal vector $N(s)$ on each $s \in \Gamma$, and then, we can define the function on Γ ,

$$f(s) = \frac{1}{(2m)!} \left(N(s) \frac{\partial}{\partial s} \right)^{2m} V(s), \quad \forall s \in \Gamma. \quad (4.4)$$

Then by (4.2) and (4.6), $f(s) > 0$, $\forall s \in \Gamma$.

Finally we assume that the function f achieves its minimum on Γ on a finite number of discrete points:

$$\Sigma_0 = f^{-1}(\{\eta_0\}) = \{s_1, \dots, s_{\ell_0}\}, \quad \text{if } \eta_0 = \min_{s \in \Gamma} f(s), \quad (4.5)$$

and the hessian of f at each point $s_j \in \Sigma_0$ is non degenerated:

$$\exists \eta_1 > 0 \text{ s.t.}$$

$$\frac{1}{2} \langle d(\langle df; w \rangle); w \rangle(s_j) \geq \eta_1 |w(s_j)|^2, \quad \forall w \in T\Gamma, \forall s_j \in \Sigma_0. \quad (4.6)$$

If $g = (g_{ij})$ is the riemannian metric on Γ , then $|w(s)| = (g(w(s), w(s)))^{1/2}$. The hessian of f at each $s_j \in \Sigma_0$, is the symmetric operator on $T_{s_j}\Gamma$, $Hess(f)_{s_j}$, associated to the two-bilinear form defined on $T_{s_j}\Gamma$ by :

$$(v, w) \in (T_{s_j}\Gamma)^2 \rightarrow \frac{1}{2} \langle d(\langle df; \tilde{v} \rangle); \tilde{w} \rangle(s_j), \quad (4.7)$$

$$\forall (\tilde{v}, \tilde{w}) \in (T\Gamma)^2 \text{ s.t. } (\tilde{v}(s_j), \tilde{w}(s_j)) = (v, w).$$

$Hess(f)_{s_j}$ has $d-1$ non negative eigenvalues

$$\rho_1^2(s_j) \leq \dots \leq \rho_{d-1}^2(s_j), \quad (\rho_j(s_j) > 0).$$

In local coordinates, those eigenvalues are the ones of the symmetric matrix

$$\frac{1}{2} G^{1/2}(s_j) \left(\frac{\partial^2}{\partial x_k \partial x_\ell} f(s_j) \right)_{1 \leq k, \ell \leq d-1} G^{1/2}(s_j), \quad (G(x) = (g_{k,\ell}(x))_{1 \leq k, \ell \leq d-1}).$$

The eigenvalues $\rho_k^2(s_j)$ do not depend on the choice of coordinates. We denote

$$Tr^+(Hess(f(s_j))) = \sum_{\ell=1}^{d-1} \rho_\ell(s_j). \quad (4.8)$$

We denote by $(\mu_j)_{j \geq 1}$ the increasing sequence of the eigenvalues of the operator $-\frac{d^2}{dt^2} + t^{2m}$ on $L^2(\mathbb{R})$,
and by $(\varphi_j(t))_{j \geq 1}$ the associated orthonormal Hilbert base of eigenfunctions.

Theorem 4.1 Under the above assumptions, for any $N \in \mathbb{N}^*$, there exist $h_0 \in]0, 1]$ and $C_0 > 0$ such that, if $\mu_j < h^{-4m/(m+1)(2m+3)}$, and if $\alpha \in \mathbb{N}^{d-1}$ and $|\alpha| \leq N$, then $\forall s_\ell \in \Sigma_0$, $\exists \lambda_{j\ell\alpha}^h \in \text{sp}_d(P^h)$ s.t.

$$\begin{aligned} & \left| \lambda_{j\ell\alpha}^h - h^{2m/(m+1)} \left[\eta_0^{1/(m+1)} \mu_j + h^{1/(m+1)} \mu_j^{1/2} \mathcal{A}_\ell(\alpha) \right] \right| \\ & \leq h^2 \mu_j^{2+3/2m} C_0 ; \end{aligned}$$

$$\begin{aligned} \text{with } \mathcal{A}_\ell(\alpha) &= \frac{1}{\eta_0^{m/(2m+2)} (m+1)^{1/2}} [2\alpha\rho(s_\ell) + \text{Tr}^+(\text{Hess}(f(s_\ell)))] . \\ (\alpha\rho(s_\ell) &= \alpha_1\rho_1(s_\ell) + \dots + \alpha_{d-1}\rho_{d-1}(s_\ell)) . \end{aligned}$$

Proof :

Let $\mathcal{O}_0 \subset \mathbb{R}^d$ be an open neighbourhood of $s_l \in \Sigma_0$, such that there exists $\phi \in C^\infty(\mathcal{O}_0; \mathbb{R})$ satisfying

$$\begin{aligned} \Gamma_0 &= \Gamma \cap \mathcal{O}_0 = \{z \in \mathcal{O}_0 ; \phi(z) = 0\} ; \\ |\nabla \phi(z)| &= 1, \quad \forall z \in \mathcal{O}_0 . \end{aligned} \tag{4.9}$$

After changing \mathcal{O}_0 into a smaller neighbourhood if necessary, we can find $\tau \in C^\infty(\mathcal{O}_0; \mathbb{R}^{d-1})$ such that $\tau(s_l) = 0$ and $\forall z \in \mathcal{O}_0$,

$$\begin{aligned} \nabla \tau_j(z) \cdot \nabla \phi(z) &= 0, \quad \forall j = 1, \dots, d-1 \\ \text{rank}\{\nabla \tau_1(z), \dots, \nabla \tau_{d-1}(z)\} &= d-1 . \end{aligned} \tag{4.10}$$

Then $(x, y) = (x_1, \dots, x_{d-1}, y) = (\tau_1, \dots, \tau_{d-1}, \phi)$ are local coordinates in \mathcal{O}_0 such that

$$\begin{aligned} \Delta &= |\tilde{g}|^{-1/2} \sum_{1 \leq i, j \leq d-1} \partial_{x_i} (|\tilde{g}|^{1/2} \tilde{g}^{ij} \partial_{x_j}) + |\tilde{g}|^{-1/2} \partial_y (|\tilde{g}|^{1/2} \partial_y) \\ V &= y^{2m} \tilde{f}(x, y) \quad \text{with } \tilde{f} \in C^\infty(\mathcal{V}_0) ; \end{aligned} \tag{4.11}$$

\mathcal{V}_0 is an open neighbourhood of zero in \mathbb{R}^d ,
 $\tilde{g}^{ij}(x, y) = \tilde{g}^{ji}(x, y) \in C^\infty(\mathcal{V}_0; \mathbb{R})$, $|\tilde{g}|^{-1} = \det(\tilde{g}^{ij}(x, y)) > 0$.

$x = (x_1, \dots, x_{d-1})$ are local coordinates on Γ_0
and the metric $g = (g_{ij})$ on Γ_0 is given by

$$(g_{ij}(x))_{1 \leq i, j \leq d-1} = G(x), \quad \text{with } (G(x))^{-1} = (\tilde{g}^{ij}(x, 0))_{1 \leq i, j \leq d-1} .$$

If $w \in C_0^2(\mathcal{O}_0)$ then

$$\begin{aligned} P^h w &= \widehat{P}^h u \quad \text{with} \\ u &= |\tilde{g}|^{1/4} w \quad \text{and} \\ \widehat{P}^h &= -h^2 \sum_{1 \leq i, j \leq d-1} \partial_{x_i} (\tilde{g}^{ij} \partial_{x_j}) - h^2 \partial_y^2 + V + h^2 V_0, \end{aligned} \quad (4.12)$$

for some $V_0 \in C^\infty(\mathcal{V}_0; \mathbb{R})$.

Let us write

$$V(x, y) = y^{2m} f(x) + y^{2m+1} f_1(x) + y^{2m+2} \tilde{f}_2(x, y) : \quad (4.13)$$

$$f(x) = \tilde{f}(x, 0) \quad \text{and} \quad \tilde{f}_2 \in C^\infty(\mathcal{V}_0).$$

We perform the change of variable (2.1) and the related unitary transformation,

$$(x, y) \rightarrow (x, t) = (x, f^{1/(2(m+1))}(x)y), \quad u \rightarrow v = f^{-1/(4(m+1))} u,$$

to get that

$$\begin{aligned} \widehat{P}^h u &= \widehat{Q}^h v \quad \text{with} \\ \widehat{Q}^h &= Q_0^h + t^{2m+1} f_1^0(x) + h^2 R_0 + +h^2 t R_1 + t^{2m+2} \tilde{f}_2^0 : \\ Q_0^h &= -h^2 \sum_{1 \leq i, j \leq d-1} \partial_{x_i} (g^{ij} \partial_{x_j}) + f^{1/(m+1)}(x) (-h^2 \partial_t^2 + t^{2m}) \end{aligned} \quad (4.14)$$

$$\text{and} \quad R_0 = ta(x, t)(\partial_x f(x) \partial_x) \partial_t + b(x, t) t \partial_t +$$

$$\sum_{ij} b_{ij}(x, t) \partial_{x_i} f(x) \partial_{x_j} f(x) (t \partial_t)^2 + c(x, t),$$

$$R_1 = \sum_{1 \leq i, j \leq d-1} \partial_{x_i} (\alpha_{ij}(x, t) \partial_{x_j}), \quad \text{all coefficients are regular in a neighbourhood of the zero in } \mathbb{R}^d.$$

Let μ_j be as in the theorem 4.1. We define $h_j = h^{1/(m+1)} / \mu_j^{1/2}$. Let \mathcal{O}'_0 be a bounded open neighbourhood of zero in \mathbb{R}^{d-1} such that $\overline{\mathcal{O}'_0} \subset \mathcal{O}_0 \cap \{(x, 0) ; x \in \mathbb{R}^{d-1}\}$.

We consider the Dirichlet operator on $L^2(\mathcal{O}'_0)$, $H_0^{h_j}$:

$$H_0^{h_j} = -h_j^2 \sum_{1 \leq k, \ell \leq d-1} \partial_{x_k} (g^{k\ell}(x) \partial_{x_\ell}) + f^{1/(m+1)}(x). \quad (4.15)$$

It is well known, (see for example [He1] or [He-Sj1], that for any $\alpha \in \mathbb{N}^{d-1}$ satisfying the assumptions of the theorem 4.1, one has:

$$\exists \lambda_{j,\alpha}^h \in \text{sp}(H_0^{h_j}) \text{ s.t. } |\lambda_{j,\alpha}^h - [\eta_0^{1/(m+1)} + h_j \mathcal{A}_l(\alpha)]| \leq h_j^2 C;$$

$\mathcal{A}_l(\alpha)$ is defined in theorem 4.1 in relation with our $s_l \in \Sigma_0$.

C is a constant depending only on N . We will denote by $\psi_{j,\alpha}^{h_j}(x)$ any associated eigenfunction with a L^2 -norm equal to 1. Let $\chi_0 \in C^\infty(\mathbb{R})$ such that

$$\chi_0(t) = 1 \text{ if } |t| \leq 1/2 \text{ and } \chi(t) = 0 \text{ if } |t| \geq 1.$$

We define the following function :

$$u_{j,\alpha}^h(x, t) = h^{-1/(2m+2)} \chi_0(t/\epsilon_0) \psi_{j,\alpha}^{h_j}(x) [\varphi_j(h^{-1/(m+1)}t) - h^{1/(m+1)} F_j^h(x, t)],$$

with

$$F_j^h(x, t) = f_1^0(x) f^{-1/(m+1)}(x) \phi_j(h^{-1/(m+1)}t),$$

where $\phi_j \in S(\mathbb{R})$ is solution of :

$$-\frac{d^2}{dt^2} \phi_j(t) + (t^{2m} - \mu_j) \phi_j(t) = t^{2m+1} \varphi_j(t),$$

and $\epsilon_0 \in]0, 1]$ is a small enough constant, but independent of h and j . ϕ_j exists because μ_j is a non-degenerated eigenvalue and the related eigenfunction φ_j (see 1.8) verifies $\int_{\mathbb{R}} t^{2m+1} \varphi_j^2(t) dt = 0$, since it is a real even or odd function.

Using the similar estimates as in chapter 3, one can get easily that

$$\mu_j^{-1} \|t \partial_t \varphi_j\|_{L^2(\mathbb{R})} + \mu_j^{-2} \|(t \partial_t)^2 \varphi_j\|_{L^2(\mathbb{R})} \leq C$$

and $\forall k \in \mathbb{N}$, $\exists C_k > 0$ s.t. $\mu_j^{-k/2m} \|t^k \varphi_j\|_{L^2(\mathbb{R})} \leq C_k$.

It is well known that there exists $\epsilon_1 > 0$ s.t.

$|\mu_j - \mu_\ell| \geq \epsilon_1$, $\forall \ell \neq j$, then the inverse of $-\frac{d^2}{dt^2} + t^{2m} - \mu_j$ is $L^2(\mathbb{R})$ -bounded by $1/\epsilon_1$, (on the orthogonal of φ_j). So in the same way as in chapter 3, we get also that

$$\mu_j^{-2-1/2m} \|t \partial_t \phi_j\|_{L^2(\mathbb{R})} + \mu_j^{-3-1/2m} \|(t \partial_t)^2 \phi_j\|_{L^2(\mathbb{R})} \leq C$$

and $\forall k \in \mathbb{N}$, $\exists C_k > 0$ s.t. $\mu_j^{-1-(k+1)/2m} \|t^k \phi_j\|_{L^2(\mathbb{R})} \leq C_k$.

As in the proof of Theorem 3.1, we get easily that

$$\|[\widehat{Q}^h - \mu_j \lambda_{j,\alpha}^h] \chi_0(|x|/\epsilon_0) u_{j,\alpha}^{h_j}(x, t)\|_{L^2(\mathcal{O}_0)} \leq h^2 \mu_j^{(4m+3)/2m} C$$

and

$$|\|\chi_0(|x|/\epsilon_0)u_{j,\alpha}^{h_j}(x,t)\|_{L^2(\mathcal{O}_0)} - 1| = \mathbf{O}(h^{1/(m+1)}\mu_j^{(2m+1)/2m}) = o(1).$$

So the theorem 4.1 follows easily.

Remark 4.2 If in Theorem 4.1 we assume that j is also bounded by N , then, as in [He-Sj4], we can get a full asymptotic expansion

$$\lambda_{j\ell\alpha}^h \sim h^{2m/(m+1)} \sum_{k=0}^{+\infty} c_{j\ell k\alpha} h^{k/(m+1)},$$

and for the related eigenfunction, a quasimode of the form

$$u_{j\ell\alpha}^h(x,t) \sim c(h)e^{-\psi(x)/h^{1/(m+1)}} \chi_0(t/\epsilon_0) \sum_{k=0}^{+\infty} h^{k/(2m+2)} a_{j\ell k\alpha}(x) \phi_{jk}(t/h^{1/(m+1)}).$$

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